

A weak Fano quadric surface bundle with a bisectonal flopping curve

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Abstract

This note gives an interesting example for smooth weak Fano 3-fold V with quadric bundle structure which has only one flopping curve. The weak Fano 3-fold V has the flop V^+ with del Pezzo fibration of degree 4. This is a counterexample of Vologodsky's result¹ [5] that says two del Pezzo fibrations differing only by a flop are of the same degree. The example has been constructed in [4].

1. Introduction

A smooth projective 3-fold V is a *weak Fano 3-fold* if its anti-canonical divisor $-K_V$ is nef and big (cf. [3]). Here a divisor $D \subset V$ is *nef* if $(D, C) \geq 0$ for any effective curve $C \subset V$, and a nef divisor D is *big* if $(D^3) > 0$. For a vector bundle \mathcal{E} of rank r over the projective line \mathbf{P}^1 , we can construct the projective bundle $\mathbf{P}(\mathcal{E}) \rightarrow \mathbf{P}^1$. Since any vector bundle over \mathbf{P}^1 can be decomposed into a direct sum of line bundles, for sake of simplicity, $\mathbf{F}(a_0, a_1, \dots, a_n)$ denotes $\mathbf{P}(\mathcal{E})$ for $\mathcal{E} = \bigoplus_{i=0}^n \mathcal{O}(a_i)$ over \mathbf{P}^1 . For example, $\mathbf{F}(0, a)$ is the Hirzebruch surface with negative section s , $s^2 = -a$, and $\mathbf{F}(0^3)$ is the direct product $\mathbf{P}^2 \times \mathbf{P}^1$.

We here construct a smooth weak Fano 3-fold V with quadric bundle structure which has a flop V^+ with del Pezzo fibration of degree 4:

$$\begin{array}{ccccccc}
 \mathbf{F}(0^2, 1^2) \supset V \supset C & \xleftarrow{\text{a flop}} & s \subset V^+ \subset \mathbf{F}(0, 1^2, 2^2) \\
 \downarrow \pi & \searrow \varphi & \swarrow \psi & \downarrow \rho \\
 \mathbf{P}^1 & & P \cap W & \mathbf{P}^1 \\
 \text{a quadric bundle} & & & \text{a del Pezzo fibration of degree 4}
 \end{array}$$

¹ Vologodsky calculated the Euler number of the rational surface containing flopped curve in [5]. Although it is possible for the surface to have singularities along the flopped curve, he treated the surface as a smooth one. This is the reason why he overlooked our case.

2. Construction of the 3-fold V

Consider the vector bundle $\mathcal{E} = \mathcal{O}^{\oplus 2} \oplus \mathcal{O}(1)^{\oplus 2}$ of rank 4 over the projective line \mathbf{P}^1 , and the \mathbf{P}^3 -bundle $\pi: X = \mathbf{P}(\mathcal{E}) = \mathbf{F}(0^2, 1^2) \rightarrow \mathbf{P}^1$. Let H and F be the tautological divisor and a fiber of $\pi: X \rightarrow \mathbf{P}^1$, respectively.

Let $V \subset X$ be a smooth 3-fold linearly equivalent to $2H + F$ as a divisor on X . The 3-fold V is a quadric surface bundle with Picard number 2, because $V \sim 2H + F$ is an ample divisor of the \mathbf{P}^3 -bundle $X = \mathbf{F}(0^2, 1^2)$ over \mathbf{P}^1 . Therefore, $\text{Pic } V$ is generated by the restrictions H_V and F_V of H and F to V , i.e., $\text{Pic } V = \mathbf{Z}[H_V] \oplus \mathbf{Z}[F_V]$, and hence $N^1(V) = \mathbf{R}[H_V] \oplus \mathbf{R}[F_V] \cong N^1(X) = \mathbf{R}[H] \oplus \mathbf{R}[F]$. By the duality, $N_1(V) \cong N_1(X) = \mathbf{R}[l] \oplus \mathbf{R}[s]$, where l and s are the dual basis with respect to H and F and they are a line in a fiber $F \cong \mathbf{P}^3$ and the minimal section of $\pi: X \rightarrow \mathbf{P}^1$, respectively. Any effective curve C in V is considered as in X , and numerically equivalent to $al + bs$ for some non-negative integers $a, b \in \mathbf{Z}_{\geq 0}$.

The adjunction formula gives $-K_V = (-K_X - V)|_V = 2H_V - F_V$. For an effective curve $C \equiv al + bs$ in V , the intersection number $(C \cdot -K_V) \leq 0$ implies $(0 \leq) 2a \leq b$. The curve C is mapped onto \mathbf{P}^1 by the projection $\pi: X \rightarrow \mathbf{P}^1$. Let $\nu: D \rightarrow C \subset X$ be the normalization of C , and $\mu = \pi \circ \nu: D \rightarrow \mathbf{P}^1$. We have $\deg \mu = b > 0$ and $\nu^* \mathcal{O}_X(1) = \mathcal{O}_D(a)$. From $\pi^* \mathcal{E} \rightarrow \mathcal{O}_X(1) \rightarrow 0$ and $\mu^* \mathcal{O}(1) = \mathcal{O}_D(b)$, it follows that

$$\pi^* \mathcal{E} = \mathcal{O}_D^{\oplus 2} \oplus \mathcal{O}_D(b)^{\oplus 2} \rightarrow \mathcal{O}_D(a) \rightarrow 0.$$

Since $a < b$, this sequence factors $\mathcal{O}_D^{\oplus 2} \rightarrow \mathcal{O}_D(a)$, hence the morphism $\nu: D \rightarrow X$ factors $D \rightarrow \mathbf{P}(\mathcal{O}^{\oplus 2}) = \mathbf{F}(0^2) \subset X$, i.e., the curve C is on the ruled surface $\mathbf{F}(0^2)$. It follows from $V \sim 2H + F$ that the intersection curve $V \cap \mathbf{F}(0^2)$ is numerically equivalent to $l + 2s$, which is no other than the curve C . Thus V has only one curve $C \equiv l + 2s$ with $(-K_V \cdot C) = 0$, which is a flopping curve (cf. [1], [2]). Moreover $-K_V$ is nef and big because

$$(-K_V)^3 = (2H_V - F_V)^3 = (2H - F)^3(2H + F) = 16(H^4 - H^3F) = 16 > 0.$$

Consequently, the smooth 3-fold $V \sim 2H + F$ in $X = \mathbf{F}(0^2, 1^2)$ is a smooth weak Fano 3-fold with Picard number 2, which has only one flopping curve $C \equiv l + 2s$ as a bisection of the quadric bundle structure $\pi: V \rightarrow \mathbf{P}^1$.

3. Construction of the flop V^+

In [4], we have already constructed the flop V^+ of this quadric bundle Fano 3-fold V . We here reconstruct more concretely by using bihomogeneous coordinates system of \mathbf{P}^3 -bundle $X = \mathbf{F}(0^2, 1^2)$.

The bihomogeneous coordinates ring of X is $R_X = \mathbf{C}[x_0, x_1, y_2, y_3, s_0, s_1]$ with bidegree $\deg x_i = (1, 0)$, $\deg y_i = (1, -1)$ and $\deg s_j = (0, 1)$. The projective space bundle X has a standard affine chart $X = \bigcup_{i=0,1,2,3; j=0,1} U_{ij}$. After making linear transformations if it is necessary, the 3-fold V is defined by the bihomogeneous polynomial

$$\begin{aligned} f(x_0, x_1, y_2, y_3, s_0, s_1) &= x_0 x_1 s_0 + (x_0^2 + x_1^2) s_1 + x_1 (l_0(y) s_0^2 + l_1(y) s_0 s_1 + l_2(y) s_1^2) \\ &\quad + q_0(y) s_0^3 + q_1(y) s_0^2 s_1 + q_2(y) s_0 s_1^2 + q_3(y) s_1^3 \\ &= f_0(x_0, x_1, y_2, y_3, s_0) s_0 + f_1(x_0, x_1, y_2, y_3, s_0, s_1) s_1 \end{aligned}$$

of $\deg f = (2, 1)$ for the linear forms $l_i(y) = l_{i0} y_2 + l_{i1} y_3$ and the quadric forms $q_i(y) = q_{i0} y_2^2 + q_{i1} y_2 y_3 + q_{i2} y_3^2$, where

$$\begin{aligned} f_0 &= x_0 x_1 + x_1 l_0(y) s_0 + q_0(y) s_0^2, & \text{and} \\ f_1 &= x_0^2 + x_1^2 + x_1 (l_1(y) s_0 + l_2(y) s_1) + (q_1(y) s_0^2 + q_2(y) s_0 s_1 + q_3(y) s_1^2) \end{aligned}$$

Let R_V be the quotient ring $R_X/(f)$.

Consider the homogeneous coordinates ring

$$R_Z = \mathbf{C}[p, a_0, a_1, b_0, b_1, c_0, c_1, c_2, d_0, d_1, d_2]$$

of \mathbf{P}^{10} . The ring homomorphisms $\varphi_j^\# : R_Z \rightarrow R_V \left[\frac{1}{s_j} \right]$ ($j=0, 1$) defined by

$$\begin{aligned} \varphi_0^\# : \quad & p \mapsto f_1(x_0, x_1, y_2, y_3, s_0, s_1)/s_0 & c_0 &\mapsto y_2^2 s_0 \\ & a_0 \mapsto x_0 y_2 & c_1 &\mapsto y_2 y_3 s_0 \\ & a_1 \mapsto x_0 y_3 & c_2 &\mapsto y_3^2 s_0 \\ & b_0 \mapsto x_1 y_2 & d_0 &\mapsto y_2^2 s_1 \\ & b_1 \mapsto x_1 y_3 & d_1 &\mapsto y_2 y_3 s_1 \\ & & d_2 &\mapsto y_3^2 s_1 \end{aligned}$$

$$\begin{array}{ll}
 p \mapsto f_0(x_0, x_1, y_2, y_3, s_0)/s_1 & c_0 \mapsto y_2^2 s_0 \\
 a_0 \mapsto x_0 y_2 & c_1 \mapsto y_2 y_3 s_0 \\
 \varphi_1^\# : a_1 \mapsto x_0 y_3 & c_2 \mapsto y_3^2 s_0 \\
 b_0 \mapsto x_1 y_2 & d_0 \mapsto y_2^2 s_1 \\
 b_1 \mapsto x_1 y_3 & d_1 \mapsto y_2 y_3 s_1 \\
 & d_2 \mapsto y_3^2 s_1
 \end{array}$$

can be glued because

$$f_1(x_0, x_1, y_2, y_3, s_0, s_1)/s_0 = f_0(x_0, x_1, y_2, y_3, s_0)/s_1 \quad \text{in} \quad R_V\left[\frac{1}{s_0}, \frac{1}{s_1}\right].$$

Let $\varphi : V \rightarrow Z$ be the morphism determined by the ring homomorphisms $\varphi_i^\#$.

The kernels of $\varphi_0^\#$ and $\varphi_1^\#$ coincide, and equal to the homogeneous ideal

$$\begin{aligned}
 I = & (d_0 p + f_0^{(0)}, d_1 p + f_0^{(1)}, d_2 p + f_0^{(2)}, c_0 p - f_1^{(1)}, c_1 p - f_1^{(1)}, c_2 p - f_1^{(2)}, \\
 & [ab], [ac_0], [ac_1], [ad_0], [ad_1], \\
 & [bc_0], [bc_1], [bd_0], [bd_1], [c_0 c_1], [c_0 d_0], [c_0 d_1], [c_1 d_0], [c_1 d_1], [d_0 d_1]).
 \end{aligned}$$

Here the polynomials $f_0^{(i)}$ and $f_1^{(i)}$ are

$$\begin{aligned}
 f_0^{(0)} &= a_0 b_0 + b_0 (l_{00} c_0 + l_{01} c_1) + (q_{00} c_0^2 + q_{01} c_0 c_1 + q_{02} c_1^2), \\
 f_0^{(1)} &= a_0 b_1 + b_1 (l_{00} c_0 + l_{01} c_1) + (q_{00} c_0 c_1 + q_{01} c_1^2 + q_{02} c_1 c_2), \\
 f_0^{(2)} &= a_1 b_1 + b_1 (l_{00} c_1 + l_{01} c_2) + (q_{00} c_1^2 + q_{01} c_1 c_2 + q_{02} c_2^2), \\
 f_1^{(0)} &= a_0^2 + b_0^2 + b_0 ((l_{10} c_0 + l_{11} c_1) + (l_{20} d_0 + l_{21} d_1)) \\
 &\quad + (q_{10} c_0^2 + q_{11} c_0 c_1 + q_{12} c_1^2) + (q_{20} c_0 d_0 + q_{21} c_0 d_1 + q_{22} c_1 d_1) \\
 &\quad + (q_{30} d_0^2 + q_{31} d_0 d_1 + q_{32} d_1^2), \\
 f_1^{(1)} &= a_0 a_1 + b_0 b_1 + b_1 ((l_{10} c_0 + l_{11} c_1) + (l_{20} d_0 + l_{21} d_1)) \\
 &\quad + (q_{10} c_0 c_1 + q_{11} c_1^2 + q_{12} c_1 c_2) + (q_{20} c_0 d_1 + q_{21} c_1 d_1 + q_{22} c_1 d_2) \\
 &\quad + (q_{30} d_0 d_1 + q_{31} d_1 d_1 + q_{32} d_1 d_2), \\
 f_1^{(2)} &= a_1^2 + b_1^2 + b_1 ((l_{10} c_1 + l_{11} c_2) + (l_{20} d_1 + l_{21} d_2)) \\
 &\quad + (q_{10} c_1^2 + q_{11} c_1 c_2 + q_{12} c_2^2) + (q_{20} c_1 d_1 + q_{21} c_1 d_2 + q_{22} c_2 d_2) \\
 &\quad + (q_{30} d_1^2 + q_{31} d_1 d_2 + q_{32} d_2^2),
 \end{aligned}$$

and the symbol $[\]$ is an abbreviation of $[ab] = a_0 b_1 - a_1 b_0$, $[ac_0] = a_0 c_1 - a_1 c_0$, $[ac_1] = a_0 c_2 - a_1 c_1$ and so on. The ideal I determines the subvariety $W \subset Z$. Denote by R_W the quotient ring R_Z/I .

We will calculate the image $\varphi(V_{ij})$ for each affine open set $V_{ij} = V \cap U_{ij}$ ($i = 0, 1, 2, 3$; $j = 0, 1$) of V . The image of

$$V_{20} = \text{Spec } \mathbf{C} \left[\frac{x_0}{y_2 s_0}, \frac{x_1}{y_2 s_0}, \frac{y_3}{y_2}, \frac{s_1}{s_0} \right] / \left(f \left(\frac{x_0}{y_2 s_0}, \frac{x_1}{y_2 s_0}, 1, \frac{y_3}{y_2}, 1, \frac{s_1}{s_0} \right) \right)$$

coincides with

$$\begin{aligned} W \cap Z_{c_0} &= W \cap \text{Spec } \mathbf{C} \left[\frac{p}{c_0}, \frac{a_0}{c_0}, \frac{a_1}{c_0}, \frac{b_0}{c_0}, \frac{b_1}{c_0}, \frac{c_1}{c_0}, \frac{c_2}{c_0}, \frac{d_0}{c_0}, \frac{d_1}{c_0}, \frac{d_2}{c_0} \right] \\ &= \text{Spec } \mathbf{C} \left[\frac{a_0}{c_0}, \frac{b_0}{c_0}, \frac{c_1}{c_0}, \frac{d_0}{c_0} \right] / \left(f_0 \left(\frac{a_0}{c_0}, \frac{b_0}{c_0}, 1, \frac{c_1}{c_0}, 1 \right) + \frac{d_0}{c_0} f_1 \left(\frac{a_0}{c_0}, \frac{b_0}{c_0}, 1, \frac{c_1}{c_0}, 1, \frac{d_0}{c_0} \right) \right) \\ &= \text{Spec } \mathbf{C} \left[\frac{a_0}{c_0}, \frac{b_0}{c_0}, \frac{c_1}{c_0}, \frac{d_0}{c_0} \right] / \left(f \left(\frac{a_0}{c_0}, \frac{b_0}{c_0}, 1, \frac{c_1}{c_0}, 1, \frac{d_0}{c_0} \right) \right), \end{aligned}$$

and the restriction map $\varphi|_{V_{20}} : V_{20} \rightarrow W_{c_0} := W \cap Z_{c_0}$ is an isomorphism. Similarly, three restriction maps $\varphi|_{V_{21}} : V_{21} \rightarrow W_{d_0}$, $\varphi|_{V_{30}} : V_{30} \rightarrow W_{c_2}$, and $\varphi|_{V_{31}} : V_{31} \rightarrow W_{d_2}$ are isomorphisms. The image $W_0 = \varphi(V_{00} \cup V_{01})$ of the union of

$$\begin{cases} V_{00} = \text{Spec } \mathbf{C} \left[\frac{x_1}{x_0}, \frac{y_2 s_0}{x_0}, \frac{y_3 s_0}{x_0}, \frac{s_1}{s_0} \right] / \left(f \left(1, \frac{x_1}{x_0}, \frac{y_2 s_0}{x_0}, \frac{y_3 s_0}{x_0}, 1, \frac{s_1}{s_0} \right) \right) \\ V_{01} = \text{Spec } \mathbf{C} \left[\frac{x_1}{x_0}, \frac{y_2 s_1}{x_0}, \frac{y_3 s_1}{x_0}, \frac{s_0}{s_1} \right] / \left(f \left(1, \frac{x_1}{x_0}, \frac{y_2 s_1}{x_0}, \frac{y_3 s_1}{x_0}, \frac{s_0}{s_1}, 1 \right) \right) \end{cases} \quad \text{and}$$

coincides with the union $W_{a_0} \cup W_{a_1} \cup P$, where

$$\begin{aligned} W_{a_0} &= W \cap \text{Spec } \mathbf{C} \left[\frac{p}{a_0}, \frac{a_1}{a_0}, \frac{b_0}{a_0}, \frac{b_1}{a_0}, \frac{c_0}{a_0}, \frac{c_1}{a_0}, \frac{c_2}{a_0}, \frac{d_0}{a_0}, \frac{d_1}{a_0}, \frac{d_2}{a_0} \right] \\ &= \text{Spec } \mathbf{C} \left[\frac{p}{a_0}, \frac{a_1}{a_0}, \frac{b_0}{a_0}, \frac{c_0}{a_0}, \frac{d_0}{a_0} \right] / \left(f_0 \left(1, \frac{b_0}{a_0}, 1, \frac{a_1}{a_0}, \frac{c_0}{a_0} \right) + \frac{p}{a_0} \frac{d_0}{a_0} \right. \\ &\quad \left. - f_1 \left(1, \frac{b_0}{a_0}, 1, \frac{a_1}{a_0}, \frac{c_0}{a_0}, \frac{d_0}{a_0} \right) - \frac{p}{a_0} \frac{c_0}{a_0} \right), \\ W_{a_1} &= W \cap \text{Spec } \mathbf{C} \left[\frac{p}{a_1}, \frac{a_0}{a_1}, \frac{b_0}{a_1}, \frac{b_1}{a_1}, \frac{c_0}{a_1}, \frac{c_1}{a_1}, \frac{c_2}{a_1}, \frac{d_0}{a_1}, \frac{d_1}{a_1}, \frac{d_2}{a_1} \right] \\ &= \text{Spec } \mathbf{C} \left[\frac{p}{a_1}, \frac{a_0}{a_1}, \frac{b_1}{a_1}, \frac{c_2}{a_1}, \frac{d_2}{a_1} \right] / \left(f_0 \left(1, \frac{b_1}{a_1}, \frac{a_0}{a_1}, 1, \frac{c_2}{a_1} \right) + \frac{p}{a_1} \frac{d_2}{a_1} \right. \\ &\quad \left. - f_1 \left(1, \frac{b_1}{a_1}, \frac{a_0}{a_1}, 1, \frac{c_2}{a_1}, \frac{d_2}{a_1} \right) - \frac{p}{a_1} \frac{c_2}{a_1} \right), \end{aligned}$$

and $P = \{[1 : 0 : \cdots : 0]\} \in W \subset Z \cong \mathbf{P}^{10}$. Indeed, one of $\frac{c_0}{a_0}$ and $\frac{d_0}{a_0}$ (resp. $\frac{c_2}{a_1}$ and $\frac{d_2}{a_1}$) is not 0 on W_{a_0} (resp. W_{a_1}), and hence the part $(y_2 \neq 0)$ (resp. $(y_3 \neq 0)$) in $V_{00} \cup V_{01}$ is isomorphic to W_{a_0} (resp. W_{a_1}) by φ and the image of $(y_2 = y_3 = 0) \subset V_{00} \cup V_{01}$ is $P \in W$. Therefore $\varphi : V_{00} \cup V_{01} \rightarrow W_{a_0} \cup W_{a_1} \cup P$ is an isomorphism outside P . Similarly, $\varphi : V_{10} \cup V_{11} \rightarrow W_{b_0} \cup W_{b_1} \cup P$ is an isomorphism outside P . The inverse image $\varphi^{-1}(P) \subset V$ of $P \in W$ is a curve $C = (y_2 = y_3 = 0) \subset V$ on $\mathbf{F}(0^2) \cong \mathbf{P}^1 \times \mathbf{P}^1$, which is the unique flopping curve on V .

Now let $\rho : Y = \mathbf{F}(0, 1^2, 2^2) \rightarrow \mathbf{P}^1$ be the \mathbf{P}^4 -bundle over \mathbf{P}^1 having bihomogeneous coordinates ring $R_Y = \mathbf{C}[u_0, v_1, v_2, w_3, w_4, t_0, t_1]$ with bidegree $\deg u_0 = (1, 0)$, $\deg v_i = (1, -1)$, $\deg w_i = (1, -2)$ and $\deg t_j = (0, 1)$. Consider two bihomogeneous polynomials

$$\begin{aligned} g_0 &= w_4 u_0 + f_0(v_1, v_2, t_0, t_1, w_3) \\ &= w_4 u_0 + (v_1 v_2 + v_2 l_0(t) w_3 + q_0(t) w_3^2), \\ g_1 &= w_3 u_0 - f_1(v_1, v_2, t_0, t_1, w_3, w_4) \\ &= w_3 u_0 - (v_1^2 + v_2^2 + v_2(l_1(t) w_3 + l_2(t) w_4) + q_1(t) w_3^2 + q_2(t) w_3 w_4 + q_3(t) w_4^2) \end{aligned}$$

of bidegree $\deg g_0 = \deg g_1 = (2, -2)$, and the subvariety $V^+ \subset Y = \mathbf{F}(0, 1^1, 2^2)$ defined by the ideal $I^+ = (g_0, g_1)$ in R_Y . The variety V^+ is an S_4 -bundle, del Pezzo fibration of degree 4, over \mathbf{P}^1 . The ring homomorphism $\psi^\# : R_Z \rightarrow R_Y$ defined by

$$\begin{array}{llll} \psi^\# : p \mapsto u_0 & a_0 \mapsto v_1 t_0 & b_0 \mapsto v_2 t_0 & c_0 \mapsto w_3 t_0^2 \quad d_0 \mapsto w_4 t_0^2 \\ & a_1 \mapsto v_1 t_1 & b_1 \mapsto v_2 t_1 & c_1 \mapsto w_3 t_0 t_1 \quad d_1 \mapsto w_4 t_0 t_1 \\ & & & c_2 \mapsto w_3 t_1^2 \quad d_2 \mapsto w_4 t_1^2 \end{array}$$

corresponds to the morphism $\psi : Y \rightarrow Z$. This morphism is defined by the linear system $|H_Y|$ on Y , where H_Y are the tautological line bundle of the \mathbf{P}^4 -bundle $\rho : Y \rightarrow \mathbf{P}^1$. Hence the morphism ψ is an isomorphism outside of the minimal section $s = \mathbf{F}(0) \subset Y \rightarrow \mathbf{P}^1$ associated to the surjection $\mathcal{O} \oplus \mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}(2)^{\oplus 2} \rightarrow \mathcal{O}$. The image $\psi(s)$ is the point $P = \{[1 : 0 : \cdots : 0]\} \in Z = \mathbf{P}^{10}$. The image $\psi(V^+) \subset Z$ is determined by the ideal

$$\begin{aligned} \psi^\#^{-1} I^+ &= (d_0 p + f_0^{(0)}, d_1 p + f_0^{(1)}, d_2 p + f_0^{(2)}, c_0 p - f_1^{(0)}, c_1 p - f_1^{(1)}, c_2 p - f_1^{(2)}, \\ &\quad [ab], [ac_0], [ac_1], [ad_0], [ad_1], \\ &\quad [bc_0], [bc_1], [bd_0], [bd_1], [c_0 c_1], [c_0 d_0], [c_0 d_1], [c_1 d_0], [c_1 d_1], [d_0 d_1]). \end{aligned}$$

This is nothing else but the ideal $I \subset R_Z$ defining the variety $W = \varphi(V) \subset Z$. The restricted morphism $\psi: V^+ \rightarrow W$ is an isomorphism outside $\psi: s \rightarrow P$.

Thus we obtain the following diagram:

$$\begin{array}{ccccc}
 \mathbf{F}(0^2, 1^2) = X \supset V \supset C & \xleftarrow{\text{a flop}} & s \subset V^+ \subset Y = \mathbf{F}(0, 1^2, 2^2) & & \\
 \downarrow \pi & \searrow \varphi & \swarrow \psi & & \downarrow \rho \\
 & P & & & \\
 & \cap & & & \\
 & W & & & \\
 & \cap & & & \\
 & Z = \mathbf{P}^{10} & & & \\
 & \nwarrow & \nearrow & & \\
 & \mathbf{P}^1 & & & \mathbf{P}^1
 \end{array}$$

and $V \dashrightarrow V^+$ is the flop corresponding $C \subset V$ to $s \subset V^+$. Here the flopping curve $C \subset V$ is a bisection of the quadric surface bundle $\pi: V \rightarrow \mathbf{P}^1$, and the flopped curve $s \subset V^+$ is a section of the del Pezzo fibration $\rho: V^+ \rightarrow \mathbf{P}^1$ of degree 4. The map φ is not defined globally on X , but is defined on V itself.

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